

Hyperinvariant subspaces of locally nilpotent linear transformations

Pudji Astuti^a, Harald K. Wimmer^{b,*}

^a*Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Bandung 40132, Indonesia*

^b*Mathematisches Institut, Universität Würzburg, 97074 Würzburg, Germany*

Abstract

A subspace X of a vector space over a field K is hyperinvariant with respect to an endomorphism f of V if it is invariant for all endomorphisms of V that commute with f . We assume that f is locally nilpotent, that is, every $x \in V$ is annihilated by some power of f , and that V is an infinite direct sum of f -cyclic subspaces. In this note we describe the lattice of hyperinvariant subspaces of V . We extend results of Fillmore, Herrero and Longstaff (Linear Algebra Appl. 17 (1977), 125–132) to infinite dimensional spaces.

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1. Introduction

Let V be a vector space over a field K and let f be an endomorphism of V . A subspace X of V is called *hyperinvariant* (with respect of f) if X is invariant for all endomorphisms of V that commute with f (see [1, p. 227], [4, p. 305]). An endomorphism f of V is said to be *locally nilpotent* [5, p. 37] if every $x \in V$ is annihilated by some power of f . In this note we are concerned with locally nilpotent endomorphisms with the property that

*Corresponding author

Email addresses: pudji@math.itb.ac.id (Pudji Astuti),
wimmer@mathematik.uni-wuerzburg.de (Harald K. Wimmer)

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the underlying vector space V is an infinite direct sum of finite-dimensional f -cyclic subspaces. It is the purpose of our paper to describe the lattice of hyperinvariant subspaces of V . We extend results of Fillmore, Herrero and Longstaff [2] (see [4, Chapter 9]) to infinite-dimensional spaces.

Let $t_j, j \in \mathbb{N}$, be the dimensions of the cyclic subspaces that are direct summands of V . Then V has a decomposition

$$V = \bigoplus_{j \in \mathbb{N}} V_{t_j}, \quad V_{t_j} = \bigoplus_{\sigma \in S_j} V_{t_j \sigma} \quad \text{where} \quad V_{t_j \sigma} \cong K[s]/f^{t_j} K[s]. \quad (1.1)$$

In (1.1) the direct summands of dimension t_j are gathered together in subspaces V_{t_j} , respectively. We assume

$$t_j < t_{j+1}, \quad j \in \mathbb{N}. \quad (1.2)$$

The main result of the paper is the following.

Theorem 1.1. *Suppose V is locally nilpotent with respect to f and let (1.1) and (1.2) hold. For a subspace $X \subseteq V$ the following statements are equivalent.*

(i) X is hyperinvariant.

(ii) X is of the form

$$X = \bigoplus_{j \in \mathbb{N}} f^{r_j} V_{t_j} \quad (1.3)$$

with

$$0 \leq r_j \leq t_j, \quad j \in \mathbb{N}, \quad (1.4)$$

and

$$r_j \leq r_\ell, \quad t_j - r_j \leq t_\ell - r_\ell, \quad \text{if} \quad j \leq \ell. \quad (1.5)$$

(iii) We have

$$X = \sum_{j \in \mathbb{N}} (\text{Im } f^{r_j} \cap \text{Ker } f^{t_j - r_j}) \quad (1.6)$$

with $(r_j)_{j \in \mathbb{N}}$ satisfying (1.4) and (1.5).

The proof of the theorem will be given in Section 2. We first introduce some notation and recall basic concepts and facts. Let $x \in V$. Define $f^0 x = x$. The smallest nonnegative integer ℓ with $f^\ell x = 0$ is called the *exponent* of x . We write $e(x) = \ell$. A nonzero vector x is said to have *height*

q if $x \in f^q V$ and $x \notin f^{q+1} V$. In this case we write $h(x) = q$. We set $h(0) = \infty$. We remark that a decomposition (1.1) implies that V has no nonzero elements of infinite height. In particular, if $x \in \bigoplus_{j=1}^r V_{t_j}$, $x \neq 0$, then $h(x) < t_r$. Let Y be a subspace of V . We set $Y[f^i] = \text{Ker}(f|_Y^i)$. Thus, in the representation (1.1) we have

$$V_{t_j}[f^{t_i}] = V_{t_j} \quad \text{if } j \leq i, \quad \text{and} \quad V_{t_j}[f^{t_i}] = f^{t_j-t_i} V_{t_j} \quad \text{if } j > i. \quad (1.7)$$

We note that the decomposition (1.1) is unique up to isomorphism. A corresponding result for abelian p -groups in [3, p. 89] shows that the cardinality of S_j is given by $\dim(f^{t_j-1} V / f^{t_j} V)[f]$. Let

$$\langle x \rangle = \text{span}\{f^i x, i \geq 0\} = \{a(f)x; a(s) \in K[s]\}$$

be the f -cyclic subspace generated by x . Adapting a definition of [3] we call a vector u a *generator* of V if $u \neq 0$ and $V = \langle u \rangle \oplus V_2$ for some invariant subspace V_2 . The set $B_u = \{u, fu, \dots, f^{e(u)-1} u\}$ is a Jordan basis of the direct summand $\langle u \rangle$ with respect to the restriction $f|_{\langle u \rangle}$. The invariant subspaces of $\langle u \rangle$ are $\langle f^q u \rangle$, $q = 0, 1, \dots, e(u)$. Given the decomposition (1.1) we choose a set of generators $\{u_{t_j \sigma}, j \in \mathbb{N}, \sigma \in S_j\}$ such that

$$V = \bigoplus_{j \in \mathbb{N}} V_{t_j}, \quad V_{t_j} = \bigoplus_{\sigma \in S_j} \langle u_{t_j \sigma} \rangle \quad \text{where} \quad e(u_{t_j \sigma}) = t_j. \quad (1.8)$$

Then $x \in V$ can be represented as a sum

$$x = \sum_{j \in \mathbb{N}, \sigma \in S_j} c_{j\sigma}(f) u_{t_j \sigma}, \quad c_{j\sigma}(s) \in K[s], \quad (1.9)$$

where only finitely many polynomials $c_{j\sigma}(s)$ are different from the zero polynomial. The representation (1.9) is unique if $\deg c_{j\sigma}(s) < e(u_{t_j \sigma}) = t_j$. Since $\{u_{t_j \sigma}; j \in \mathbb{N}, \sigma \in S_j\}$ is a basis of V we have

$$e(x) = \max\{e(c_{j\sigma}(f) u_{t_j \sigma}); c_{j\sigma}(f) u_{t_j \sigma} \neq 0\}. \quad (1.10)$$

Let $\pi_{t_j \sigma}$ denote the projection of V on $\langle u_{t_j \sigma} \rangle$ corresponding to (1.8). Thus, if x is represented by (1.9) then $\pi_{t_j \sigma} x = c_{j\sigma}(f) u_{t_j \sigma}$. Let $\text{End}(V)$ and $\text{Aut}(V)$ be the endomorphism ring and the automorphism group of V , respectively. If $\eta \in \text{End}(V)$ then $e(\eta x) \leq e(x)$. Suppose

$$\eta u_{t_j \sigma} = \sum_{\ell \in \mathbb{N}, \tau \in S_\ell} b_{\ell\tau}(f) u_{t_\ell \tau}, \quad b_{\ell\tau}(s) \in K[s]. \quad (1.11)$$

Then (1.10) implies $e(b_{\ell\tau}(f)u_{t_{\ell\tau}}) \leq e(u_{t_j\sigma}) = t_j$ if $b_{\ell\tau}(f)u_{t_{\ell\tau}} \neq 0$. Hence $f^{t_j}b_{\ell\tau}(f)u_{t_{\ell\tau}} = 0$. From $e(u_{\ell\tau}) = t_\ell$ follows $s^{t_\ell} \mid s^{t_j}b_{\ell\tau}(s)$. Hence, if $\ell > j$ then $b_{\ell\tau}(s) = s^{t_\ell - t_j}a_{\ell\tau}(s)$ for some $a_{\ell\tau}(s) \in K[s]$.

2. Proof of Theorem 1.1

Proof. (i) \Rightarrow (ii) Let V be given by (1.8) and (1.2). Suppose $X \subseteq V$ is hyperinvariant. Set $X_{t_j\sigma} = \pi_{t_j\sigma}X$. Then

$$X \cap \langle u_{t_j\sigma} \rangle \subseteq X_{t_j\sigma} \subseteq \langle u_{t_j\sigma} \rangle.$$

Because of $\pi_{t_j\sigma} \in \text{End}(V)$ we have $X_{t_j\sigma} \subseteq X$. Hence $X_{t_j\sigma} = X \cap \langle u_{t_j\sigma} \rangle$, and therefore $X_{t_j\sigma}$ is an invariant subspace contained in $\langle u_{t_j\sigma} \rangle$, and we conclude that

$$X_{t_j\sigma} = \langle f^{r_{j\sigma}}u_{t_j\sigma} \rangle \quad \text{for some } r_{j\sigma} \text{ with } 0 \leq r_{j\sigma} \leq t_j.$$

Then

$$X \subseteq \bigoplus_{j \in \mathbb{N}, \sigma \in S_j} (\pi_{t_j\sigma}X) = \bigoplus_{j \in \mathbb{N}, \sigma \in S_j} (X \cap \langle u_{t_j\sigma} \rangle) = \bigoplus_{j \in \mathbb{N}, \sigma \in S_j} \langle f^{r_{j\sigma}}u_{t_j\sigma} \rangle \subseteq X$$

implies

$$X = \bigoplus_{j \in \mathbb{N}} X_{t_j} \quad \text{with} \quad X_{t_j} = X \cap V_{t_j} = \bigoplus_{\sigma \in S_j} \langle f^{r_{j\sigma}}u_{t_j\sigma} \rangle, \quad j \in \mathbb{N}. \quad (2.1)$$

Let $\alpha_{\sigma\rho}^{(j)} \in \text{Aut}(A)$ be the automorphism with exchanges the generators $u_{t_j\sigma}$ and $u_{t_j\rho}$ in the sense that

$$\alpha_{\sigma\rho}^{(j)}(u_{t_j\sigma}, u_{t_j\rho}) = (u_{t_j\rho}, u_{t_j\sigma}), \quad \text{and} \quad \alpha_{\sigma\rho}^{(j)}(u_{t_j\lambda}) = u_{t_j\lambda} \quad \text{if } \lambda \notin \{\rho, \sigma\}.$$

Then $\alpha_{\sigma\rho}^{(j)}X = X$ and $\alpha_{\sigma\rho}^{(j)}X_{t_j} = X_{t_j}$. Therefore (2.1) yields $r_{j\sigma} = r_{j\rho}$ for $\sigma, \rho \in S_j$. Hence

$$X_{t_j} = f^{r_j}V_{t_j} \quad \text{for some } r_j \text{ with } 0 \leq r_j \leq t_j. \quad (2.2)$$

To show that inequalities (1.5) are satisfied we use suitable endomorphisms $\eta_k^\ell \in \text{End}(V)$. Let $j, k, \ell \in \mathbb{N}$, $\tau \in S_j$. For each ℓ we fix an element $\hat{\mu}_\ell$ of S_ℓ . Let η_k^ℓ be defined on the set of generators $\{u_{t_j\tau}\}$ of V by

$$\begin{aligned} \eta_k^\ell u_{t_j\tau} &= 0 && \text{if } j \neq k, \\ \eta_k^\ell u_{t_k\tau} &= \begin{cases} u_{t_\ell\hat{\mu}_\ell} & \text{if } k > \ell \\ f^{t_\ell-t_k} u_{t_\ell\hat{\mu}_\ell} & \text{if } k < \ell. \end{cases} \end{aligned}$$

Then

$$\text{Im } \eta_k^\ell = \langle u_{t_\ell\hat{\mu}_\ell} \rangle [f^{t_k}] \quad \text{and} \quad \text{Ker } \eta_k^\ell = \bigoplus_{j \in \mathbb{N}, j \neq k} V_{t_j}.$$

Hence (2.1) and (2.2) imply

$$\eta_k^\ell X = \eta_k^\ell X_{t_k} = \eta_k^\ell f^{r_k} V_{t_k}.$$

If $k > \ell$ then

$$\eta_k^\ell f^{r_k} V_{t_k} = \langle f^{r_k} u_{t_\ell\hat{\mu}_\ell} \rangle \subseteq V_{t_\ell} \cap X = X_{t_\ell} = f^{r_\ell} V_{t_\ell} = \bigoplus_{\tau \in S_\ell} \langle f^{r_\ell} u_{\ell\tau} \rangle.$$

Hence $\langle f^{r_k} u_{t_\ell\hat{\mu}_\ell} \rangle \subseteq \langle f^{r_\ell} u_{t_\ell\hat{\mu}_\ell} \rangle$, and we obtain $r_k \geq r_\ell$. If $k < \ell$ then

$$\eta_k^\ell f^{r_k} V_{t_k} = \langle f^{r_k} f^{t_\ell-t_k} u_{t_\ell\hat{\mu}_\ell} \rangle \subseteq X_{t_\ell} = f^{r_\ell} V_{t_\ell}$$

implies $\langle f^{r_k+t_\ell-t_k} u_{t_\ell\hat{\mu}_\ell} \rangle \subseteq \langle f^{r_\ell} u_{t_\ell\hat{\mu}_\ell} \rangle$. Thus we obtain $r_k + t_\ell - t_k \geq r_\ell$, that is, $t_\ell - r_\ell \geq t_k - r_k$.

(ii) \Rightarrow (iii) Let X be a subspace of the form (1.3) with (1.4) and (1.5). From $0 \leq r_j \leq t_j$ follows $f^{r_j} V_{t_j} = V_{t_j} [f^{t_j-r_j}] \subseteq V [f^{t_j-r_j}]$. Therefore $f^{r_j} V_{t_j} \subseteq f^{r_j} V \cap V [f^{t_j-r_j}]$, which implies $X \subseteq \sum_{j \in \mathbb{N}} (f^{r_j} V \cap V [f^{t_j-r_j}])$.

Now let $x \in f^{r_j} V \cap V [f^{t_j-r_j}]$, $x \neq 0$. Then $x = f^{r_j} y$ for some nonzero $y \in V$ with $f^{t_j} y = 0$. Let y be decomposed in accordance with (1.3) such that

$$y = y_1 + \cdots + y_n \quad \text{with} \quad y_\nu \in V_{t_{j_\nu}}, \quad y_\nu \neq 0, \quad \nu = 1, \dots, n.$$

Then $\max \{e(y_\nu); \nu = 1, \dots, n\} = e(y) \leq t_j$, and therefore

$$e(f^{r_j} y_\nu) \leq t_j - r_j. \tag{2.3}$$

We have $x \in X$ if $f^{r_j} y_\nu \in X$, $\nu = 1, \dots, n$. Let $j_\nu \leq j$. Then $r_{j_\nu} \leq r_j$, and therefore

$$f^{r_j} y_\nu \in f^{r_j} V_{t_{j_\nu}} \subseteq f^{r_{j_\nu}} V_{t_{j_\nu}} \subseteq X.$$

Let $j \leq j_\nu$. Then $t_j - r_j \leq t_{j_\nu} - r_{j_\nu}$ and (2.3) imply

$$f^{r_j} y_\nu \in V_{t_{j_\nu}} [f^{t_j - r_j}] \subseteq V_{t_{j_\nu}} [f^{t_{j_\nu} - r_{j_\nu}}] = f^{r_{j_\nu}} V_{t_{j_\nu}} \subseteq X.$$

Thus we have shown that $x = \sum_{\nu=1}^n f^{r_j} y_\nu \in X$, which proves the inclusion

$$f^{r_j} V \cap V[f^{t_j - r_j}] \subseteq X \quad \text{for all } j \in \mathbb{N}.$$

(iii) \Rightarrow (i) This is obvious, since $\text{Im } f^p$ and $\text{Ker } f^q$ are hyperinvariant subspaces for all $p, q \in \mathbb{N}_0$. \square

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